Show that any binary relation that is both reflexive and Euclidean is transitive, i.e., show that

$$\forall x Rxx, (\forall x, y, z)(Rxy \wedge Rxz \rightarrow Ryz) \vdash (\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz).$$

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \land Rxz \to Ryz)$	Prem
(14)	$(\forall x, y, z)(Rxy \land Ryz \to Rxz)$	$\forall$Int$\times$3 14

As always, first write down your premises, leave some blank space, and write down your conclusion.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Next, look at the conclusion. What is the main operator(s)? What strategy does it suggest?

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Since it is a sequence of three $\forall$s, we will reach 10 by using $\forall$Int$\times$3.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \to Ryz)$	Prem
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \to Rxz)$	$\forall$Int$\times$3 14

As a result, what formula should be taken as subgoal?

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Notice that the choice of a, b, c doesn't matter, as long as the two conditions on $\forall$Int (remember?) are met.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Now, this subgoal is a conditional. So our sub-subgoal is a CP.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	Rac	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

As always, for CP, we write the antecedent of the subgoal on the first line of the subproof, and the consequence on the last line.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \land Rxz \to Ryz)$	Prem
(3)	$\quad Rab \land Rbc$	Supp
	$\quad Rac$	
(13)	$Rab \land Rbc \to Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \land Ryz \to Rxz)$	$\forall$Int$\times$3 14

Here's a good trick for such deductions: Make *Rac* match the consequent of the second premise, to reach our sub-subgoal by MP.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \to Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	$R\_a \wedge R\_c \to Rac$	
	Rac	
(13)	$Rab \wedge Rbc \to Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \to Rxz)$	$\forall$Int×3 14

That means we'll instantiate y with a and z with c in line 2.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	$R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	Rac	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

But what will we instantiate x with? We'll try to match information we've already got. b would be a good guess.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	$Rba \wedge Rbc$	
	$R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	Rac	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Let's try to deduce the antecedent to set up our MP.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$\quad Rab \wedge Rbc$	Supp
	$\quad Rba$	
	$\quad Rbc$	
	$\quad Rba \wedge Rbc$	
	$\quad R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	$\quad Rac$	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Rbc is easy to get. But how will we obtain *Rba*? We'll **again** use the same trick we used below.

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \to Ryz)$	Prem
(3)	$\quad Rab \wedge Rbc$	Supp
	$\quad R\_b \wedge R\_a \to Rba$	
	$\quad Rba$	
	$\quad Rbc$	
	$\quad Rba \wedge Rbc$	
	$\quad R\underline{b}a \wedge R\underline{b}c \to Rac$	
	$\quad Rac$	
(13)	$Rab \wedge Rbc \to Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \to Rxz)$	$\forall$Int$\times$3 14

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	$R\underline{a}b \wedge R\underline{a}a \rightarrow Rba$	
	Rba	
	Rbc	
	$Rba \wedge Rbc$	
	$R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	Rac	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

Do not forget that you can instantiate two variables with *a*!

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$Rab \wedge Rbc$	Supp
	$Rab \wedge Raa$	
	$R\underline{a}b \wedge R\underline{a}a \rightarrow Rba$	
	Rba	
	Rbc	
	$Rba \wedge Rbc$	
	$R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	Rac	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$\quad Rab \wedge Rbc$	Supp
(4)	$\quad Rab$	$\wedge$Elim
	$\quad Rab \wedge Raa$	
	$\quad R\underline{a}b \wedge R\underline{a}a \rightarrow Rba$	
	$\quad Rba$	
	$\quad Rbc$	
	$\quad Rba \wedge Rbc$	
	$\quad R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	$\quad Rac$	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \wedge Rxz \rightarrow Ryz)$	Prem
(3)	$\quad Rab \wedge Rbc$	Supp
(4)	$\quad Rab$	$\wedge$Elim
(5)	$\quad Raa$	$\forall$Elim 1
	$\quad Rab \wedge Raa$	
	$\quad R\underline{a}b \wedge R\underline{a}a \rightarrow Rba$	
	$\quad Rba$	
	$\quad Rbc$	
	$\quad Rba \wedge Rbc$	
	$\quad R\underline{b}a \wedge R\underline{b}c \rightarrow Rac$	
	$\quad Rac$	
(13)	$Rab \wedge Rbc \rightarrow Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \wedge Ryz \rightarrow Rxz)$	$\forall$Int$\times$3 14

(1)	$\forall x Rxx$	Prem
(2)	$\forall x, y, z(Rxy \land Rxz \to Ryz)$	Prem
(3)	$Rab \land Rbc$	Supp
(4)	Rab	$\land$Elim
(5)	Raa	$\forall$Elim 1
(6)	$Rab \land Raa$	$\land$Int 4,5
(7)	$R\underline{a}b \land R\underline{a}a \to Rba$	$\forall$Elim$\times$3 2
(8)	Rba	MP 7,6
(9)	Rbc	$\land$Elim 3
(10)	$Rba \land Rbc$	$\land$Int 8,9
(11)	$R\underline{b}a \land R\underline{b}c \to Rac$	$\forall$Elim$\times$3 2
(12)	Rac	MP 11,10
(13)	$Rab \land Rbc \to Rac$	CP 3-12
(14)	$(\forall x, y, z)(Rxy \land Ryz \to Rxz)$	$\forall$Int$\times$3 14